Solving the Bi-Objective Portfolio Optimization Problem with Uncertain Market Parameters: A Comparative Study

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Abstract. In this paper, we study the bi-objective portfolio optimization problem with uncertain market parameters. We establish a bi-objective portfolio optimization model where both returns and covariance matrix are uncertain and propose two population based-methods for approximately solving the resulting model. A comparative study is provided for analyzing the robustness of both considered methods. Their performances are evaluated according to the hypervolume value, which is used for measuring the density of the Pareto front achieved.

Keywords: Bi-objective, heuristics, hypervolume, portfolio, robustness, worst-case

1 Introduction

The portfolio optimization was initially studied by Markowitz [11], where a quantified approach has been used for evaluating the trade-off between the expected return and the risk of portfolio of financial assets. A simple mathematical model for portfolio selection was proposed in Markowitz [11, 12], where its portfolio selection is called mean-variance model: the return on a portfolio is measured by the expected value of the random portfolio return, and the associated risk is quantified by the variance of the portfolio return.

Despite the theoretical success of the Markowitz's mean-variance model (Markowitz [12]), practitioners had less used that model. Indeed, in practice, it has been remarked that the Markowitz efficiency is related to an error-prone procedure which often results in error-maximized and makes irrelevant investment portfolios (Michaud [13]). Such a behavior is a reflection of the fact that solutions are sensitive to perturbations of the problem' parameters, because the the market parameters' estimations are subject to statistical errors and so, the results of the subsequent optimization remain unreliable (Chopra [3]).

Generally, the Markowitz model is criticized as less efficient with axiomatic models of preferences for choice under risk. In Levy [10] the author affirmed that models with regard to the preferences are based on the relation of stochastic dominance or on the expected utility theory. For that reason, Ballestero and Romero [2] suggested maximizing the investor expected utility of returns over the efficient frontier.

Several studies-based techniques have been suggested to reduce the sensitivity of the Markowitz-optimal portfolios to input uncertainty. Indeed, a first technique has been proposed by Chopra [4] and Frost and Savarino [7], where they suggested to constrain the portfolio weights. In fact, Chopra [4] proposed the use of the James-Stein estimator for evaluating the means while Frost and Savarino [7] proposed the use of Bayesian estimation of means and covariances.

These techniques reduce the sensitivity of the portfolio composition to the parameter estimates, but they are not able to provide any guarantees on the risk-return performance of the portfolio. Recently scenario-based stochastic programming models have also been proposed for handling the uncertainty in parameters. All approaches cited above do not provide any hard guarantees on the portfolio performance and become very inefficient as the number of assets grows.

In this paper we propose alternative deterministic models that are robust to parameters uncertainty and estimation errors (Markowitz [11]) and how to formulate and solve the bi-objective portfolio optimization problems under uncertainty. The perturbations in the market parameters are modeled as unknown, bounded, where the worst case criterion related to these perturbations. In order to achieve the robust efficient frontier, we propose to two alternative solutions which are based upon population- based heuristics: (i) NSGA-II (Non-Dominated Sorting Genetic Algorithm − Klalyanmoy [8, 9]) and (ii) SPEA2 (Strength Pareto Evolutionary Algorithm − Eckart *et al.* [5]).

2 Bi-objective portfolio optimization under uncertainty

Let n be the assets (the cardinality of the set $I = \{1, \dots, n\}$) with random rates of returns $\xi_1, \dots, \xi_n$. The rates of the returns to determine are $E[\xi_1], \dots, E[\xi_n]$. We set $\mu_i = E[\xi_i]$. The risk can be used to represent what is unknown about a portfolio, which is generally represented by the variance of the portfolio, denoted σ^2. In this case, the variance of a random variable is represented by $\sigma^2 = E(\xi^2) - (\bar{\xi})^2$. The variance of the portfolio can be computed using the covariance σ_{ij} between assets i and j.

2.1 *A deterministic model*

Several ways can be used for determining the risk, such as *value at risk*, *variance covariance approach* and *conditional value at risk* (Alexander *et al.* [1]). The standard bi-objective deterministic portfolio programme can be stated as follows:

$$(P_{Dp}) : \max \sum_{i=1}^{n} \mu_i x_i \ ; \ \min \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j, \ \text{s.t.} \quad \sum_{i=1}^{n} x_i = 1, \ x_i \geq 0, \ i \in I.$$

Of course, other additional objective functions have been considered in literature, like surplus variance, cardinality, portfolio value at risk, annual dividend and asset ranking (Ehrgott *et al.* [6] and Subbu *et al.* [14]). These objective functions often depend on the investors type of demand.

2.2 *Using uncertain parameters*

Assume that the expected return vector μ_i, $i = 1, \ldots, n$, and the covariance matrix σ_{ij} may take the form of intervals:

$U_\mu = \{\mu \mid \underline{\mu} \leq \mu \leq \overline{\mu}\}$ and $U_{\sigma_{ij}} = \{\sigma_{ij} \mid \underline{\sigma}_{ij} \leq \sigma_{ij} \leq \overline{\sigma}_{ij}\}, \sigma_{ij} \geq 0,\ i \in I,\ j \in I,$

where $\underline{\mu}$, $\overline{\mu}$, $\underline{\sigma}_{ij}$ and $\overline{\sigma}_{ij}$ denote both lower and upper bounds of the intervals. U_μ and $U_{\sigma_{ij}}$ are the respective sets of uncertainty characterizing μ and σ. The restriction $\sigma_{ij} \geq 0$ indicates that σ is a symmetric positive semi-definite matrix which is a necessary property for the uncertain σ_{ij} to be a covariance matrix.

Hence, the uncertain bi-objective portfolio optimization is given as follows:

$$(P_{Up}) \begin{cases} \max f_1(x) = \sum\limits_{i=1}^{n} \mu_i x_i \\ \min f_2(x) = \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} \sigma_{ij} x_i x_j \\ \text{s.t.} \qquad\qquad \sum\limits_{i=1}^{n} x_i = 1 \\ \qquad\qquad\qquad x_i \geq 0, i \in I, \end{cases} \tag{1}$$

where $\mu_i \in [\underline{\mu}_i, \overline{\mu}_i]$ and $\sigma_{ij} \in [\underline{\sigma}_{ij}, \overline{\sigma}_{ij}], i \in I, j \in I$.

Herein, we propose to extend the model (P_{Up}) (described above) by introducing the worst-case criterion. Indeed, the robustness of the strategy can be guaranteed by using a *min-max formulation* to ensure optimal *worst-case performance*.
Let U_μ and $U_{\sigma_{ij}}$ be the respective uncertainty sets associated to μ and σ. The worst-case return can be formulated as follows:

$$\text{worst-case return} = \min \sum_{i=1}^{n} \mu_i x_i,\ x_i \geq 0,\ i \in I.$$

By using the min-max representation described above, the worst-case robust multi-objective portfolio optimization formulation (denoted $P_{U\omega}$) can be written as follows:

$$(P_{U\omega}) \begin{cases} \max\limits_{x \in X} \min\limits_{\mu_i \in U_\mu} f_1(x) = \sum\limits_{i=1}^{n} \mu_i x_i \\ \min\limits_{x \in X} \max\limits_{\sigma_{ij} \in U_{\sigma_{ij}}} f_2(x) = \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} \sigma_{ij} x_i x_j \\ \text{s.t.} \qquad\qquad \sum\limits_{i=1}^{n} x_i = 1 \\ \qquad\qquad\qquad x_i \geq 0, i \in I. \end{cases} \tag{2}$$

where $\mu_i \in [\underline{\mu}_i, \overline{\mu}_i]$ and $\sigma_{ij} \in [\underline{\sigma}_{ij}, \overline{\sigma}_{ij}]$ for $i \in I$ and $j \in I$.

2.3 *Efficient frontier*

The aim is to graphically represent the so-called Markowitz frontier, when all the optimum portfolios are achieved. In this case, all solutions are plotted on a risk-returns graph for illustrating the efficient frontier. The solutions of the efficient frontier represent the portfolios that achieve the maximum returns for a current level of risk. One can observa that the efficient frontier, can be built by using a three-phase procedure:

1. First, the minimum-risk problem, without any return constraint is solved; thus a minimum expected return μ_{min} is reached.
2. Second, the maximum-returns problem, without any constraint on the level of risk, is solved; thus a maximum expected return μ_{max} is reached.
3. Third and last, the minimum-risk problem for a set of points $\mu \in [\mu_{min}, \mu_{max}]$ is solved and so, the resulting set of points forms the efficient frontier of the problem.

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